POINT MEASUREMENTS FOR A NEUMANN-TO-DIRICHLET MAP AND THE CALDERÓN PROBLEM IN THE PLANE

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Abstract. This work considers properties of the Neumann-to-Dirichlet map for the conductivity equation under the assumption that the conductivity is identically one close to the boundary of the examined smooth, bounded and simply connected domain. It is demonstrated that the so-called bisweep data, i.e., the (relative) potential differences between two boundary points when delta currents of opposite signs are applied at the very same points, uniquely determine the whole Neumann-to-Dirichlet map. In two dimensions, the bisweep data extend as a holomorphic function of two variables to some (interior) neighborhood of the product boundary. It follows that the whole Neumann-to-Dirichlet map is characterized by the derivatives of the bisweep data at an arbitrary point. On the diagonal of the product boundary, these derivatives can be given with the help of the derivatives of the (relative) boundary potentials at some fixed point caused by the distributional current densities supported at the same point, and thus such point measurements uniquely define the Neumann-to-Dirichlet map. This observation also leads to a new, truly local uniqueness result for the so-called Calderón inverse conductivity problem.

1. Introduction.

In this work, we consider properties of the Neumann-to-Dirichlet operator, i.e., the current-to-voltage boundary map for the conductivity equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D \tag{1}$$

assuming the conductivity σ is identically one in some interior neighborhood of the boundary of the smooth, simply connected and bounded domain $D \subset \mathbb{R}^n$, $n \geq 2$. In particular, we are interested in what kind of 'point measurements' uniquely characterize the Neumann-to-Dirichlet map; as a by-product, we will obtain a new result for the two-dimensional Calderón problem with partial data.

Our main analytic tool is the so-called bisweep data, which are the (relative) potential differences between two points on ∂D when delta distribution current is driven between the very same points (cf. [11]). We demonstrate that the bisweep data uniquely determine the whole Neumann-to-Dirichlet map for any symmetric anisotropic $\sigma \in L^\infty(D, \mathbb{R}^{n\times n})$, $\sigma \geq cI > 0$, conductivity as long as it satisfies the isotropic homogeneity assumption near the object boundary.

The completeness of the bisweep data has remarkable consequences in the two-dimensional case, when D can be identified with a part of the complex plane.

2010 *Mathematics Subject Classification.* 35R30, 35Q60.

Key words and phrases. Calderón problem, Neumann-to-Dirichlet map, point measurements, (bi)sweep data, partial data.

This work was supported by the Academy of Finland (the Centre of Excellence in Inverse Problems Research and project 135979).

By extending an argument in [11, 12, 14, 16], the bisweep data can be continued as a complex analytic function of two variables to some (interior) neighborhood of $\partial D \times \partial D \subset \mathbb{C}^2$. In particular, all bisweep data are determined by the corresponding derivatives at any fixed point (z_1, z_2) on $\partial D \times \partial D$. In case $z_1 = z_2 = z$, these derivatives can be presented with the help of the derivatives at z of the relative boundary potentials caused by the current densities that are the derivatives of the delta distribution located at z. In other words, sampling the relative potentials originating from the distributional currents supported at a fixed $z \in \partial D$ by the very same distributions determines the whole Neumann-to-Dirichlet map. It also follows that the Neumann-to-Dirichlet map can be recovered from the bisweep data on any countably infinite set $\Gamma \times \Gamma \subset \partial D \times \partial D$ with an accumulation point at (z, z). (A related result that assumes continuum measurements but also provides a stability estimate can be found in [1].) On the other hand, it is known that the Neumann-to-Dirichlet map uniquely defines an isotropic L^∞-conductivity in two dimensions, as demonstrated in [2].

The problem of determining an isotropic conductivity in (1) from information on the Cauchy data of the corresponding solutions is called the Calderón problem. It was proposed by Calderón in [7] and tackled by many renowned mathematicians since. In dimensions $n \geq 3$, the first global uniqueness result for $\mathscr{C}^2$-conductivities was proven in [31], and extended for less regular conductivities in [4, 26]. In two dimensions the first global uniqueness result was provided by [25] for $\mathscr{C}^2$-conductivities. Subsequently, the regularity assumptions were relaxed in [5] and, in particular, [2] proved uniqueness for general isotropic L^∞-conductivities.

All the above mentioned articles assume that the Cauchy data are known on all of ∂D, but there also exist several results considering the partial data problem of having access only to some subset(s) of ∂D. To the best of the authors' knowledge, the most general result for the partial data case in our two-dimensional setting for the Calderón problem is currently found in [17], where it is shown that an isotropic conductivity of smoothness $\mathscr{C}^{4,\alpha}$, $\alpha > 0$, is uniquely defined by the Dirichlet-to-Neumann map restricted to any open nonempty subset of ∂D. Compared to this result, the amount of data needed for our uniqueness theorem is considerably smaller: all derivatives of the (relative) boundary potentials at a single point, caused by distributional current densities supported at the same point, suffice. Moreover, our result allows L^∞-conductivities, but only under the important and arguably rather restrictive assumption of homogeneity of σ in some interior neighborhood of ∂D. For other results on the Calderón problem with partial data, we refer to [6, 10, 18, 19, 20, 21, 22] and the references therein.

There also exists a vast literature on the (non)unique solvability of the Calderón problem for anisotropic conductivities; see, e.g., [3, 17, 23, 25, 29, 30]. In two dimensions it has been shown that an anisotropic L^∞-conductivity is determined by the Dirichlet-to-Neumann map up to a natural obstruction, i.e., up to a pushforward by an H^1-diffeomorphism that fixes the object boundary [3]; see [30] for the original ideas behind such results. For the case of partial data, the most general result is arguably found in [17], where it is shown that an anisotropic conductivity of the class $\mathscr{C}^{7,\alpha}$, $\alpha > 0$, is defined up to the natural obstruction by the Dirichlet-to-Neumann map restricted to any open nonempty subset of ∂D. The results presented in this work demonstrate that under the assumption of isotropic homogeneity close to the

object boundary, an anisotropic L^∞-conductivity is defined up to the natural obstruction by the above described point measurements for the Neumann-to-Dirichlet map.

The main reason for choosing to work with the Neumann-to-Dirichlet map instead of the Dirichlet-to-Neumann map is that in practical *electrical impedance tomography*, which is the imaging modality corresponding to the Calderón problem, the natural boundary condition on nonaccessible parts of the object boundary is the homogeneous Neumann, not the homogeneous Dirichlet condition [8, 9, 28]. In particular, the bisweep data can be approximated by real-life measurements performed by two small movable electrodes [11, 13, 16].

This text is organized as follows. In Section 2 we introduce our setting and formulate the main results. Subsequently, Section 3 provides the corresponding proofs: In Section 3.1 we introduce a useful factorization for the relative Neumann-to-Dirichlet map, Sections 3.2 and 3.3 prove the complex analytic extension property for the bisweep data, and finally the actual proofs are formulated in Section 3.4.

2. The setting and main results

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a simply connected and bounded domain with a $\mathscr{C}^\infty$-boundary. Assume that the symmetric conductivity $\sigma \in L^\infty(D, \mathbb{R}^{n\times n})$ satisfies

$$\sigma \geq cI \quad \text{for } c > 0 \qquad \text{and} \qquad \Sigma := \operatorname{supp}(\sigma - I) \text{ is a compact subset of } D,$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and the first condition is to be understood in the sense of positive definiteness almost everywhere. Our main result for the Calderón problem will be formulated only for the two-dimensional case, but some interesting intermediate theorems are valid independently of the spatial dimension.

Consider the Neumann boundary value problem

$$\nabla \cdot (\sigma \nabla u) \, = \, 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} \, = \, f \quad \text{on } \partial D \tag{2}$$

for a current density f in

$$H^s_\diamond(\partial D) \, = \, \{ g \in H^s(\partial D) \; : \; \langle g, 1 \rangle_{\partial D} = 0 \}, \tag{3}$$

with some $s \in \mathbb{R}$. Here and in what follows, ν denotes the exterior unit normal field of the respective domain, and we note that the dual of $H^s_\diamond(\partial D)$ is realized by

$$H^{-s}(\partial D)/\mathbb{C} := H^{-s}(\partial D)/\operatorname{span}\{1\}, \qquad s \in \mathbb{R}. \tag{4}$$

It follows from standard theory for elliptic boundary value problems that (2) has a unique solution u_σ in $(H^{\min\{1, s+3/2\}}(D) \cap H^1_{\mathrm{loc}}(D))/\mathbb{C}$ and that the corresponding Neumann-to-Dirichlet map

$$\Lambda_\sigma : f \mapsto u_\sigma|_{\partial D}, \quad H^s_\diamond(\partial D) \to H^{s+1}(\partial D)/\mathbb{C} \tag{5}$$

is well defined and bounded for any $s \in \mathbb{R}$ (cf., e.g., [14, 24]). We denote by $\mathbb{1} \in L^\infty(D)$ the homogeneous unit conductivity and note that the relative Neumann-to-Dirichlet map

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} : f \mapsto (u_\sigma - u_{\mathbb{1}})|_{\partial D}, \quad \mathscr{D}'_\diamond(\partial D) \to \mathscr{D}(\partial D)/\mathbb{C}$$

is well defined (and bounded as an operator from $H^s_\diamond(\partial D)$ to $H^{-s}(\partial D)/\mathbb{C}$ for any $s \in \mathbb{R}$). Here, the mean-free distributions $\mathscr{D}'_\diamond(\partial D)$ and the quotient space of smooth boundary potentials $\mathscr{D}(\partial D)/\mathbb{C}$ are defined in accordance with (3) and (4); we also use similar notations, $\mathscr{D}'(\partial D)/\mathbb{C}$ and $\mathscr{D}_\diamond(\partial D)$, when the roles of distributions and

smooth test functions are reversed. This regularity result can be deduced from standard elliptic theory (cf., e.g., [14, 24]), and it also follows from Λ_σ and $\Lambda_{\mathbb{1}}$ being pseudodifferential operators with the same symbol because σ and $\mathbb{1}$ coincide in some interior neighborhood of ∂D [23].

We define the bisweep data as the function

$$\varsigma_\sigma : (x, y) \mapsto \langle(\delta_x - \delta_y), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)\rangle_{\partial D}\,, \quad \partial D \times \partial D \to \mathbb{R}, \tag{6}$$

where δ_z denotes the delta distribution at z on ∂D. This is a generalization of the concept of (standard) sweep data from [11], or the other way around, sweep data are the restriction of bisweep data onto $\partial D \times \{y_0\}$ for some fixed $y_0 \in \partial D$. What is more, the bisweep data can be approximated by two-electrode measurements in the framework of the realistic *complete electrode model* [13].

Our first result shows that the bisweep data carry the same information as the whole (relative) Neumann-to-Dirichlet map; the proof is based on a simple polarization identity.

Theorem 2.1. *Let the above assumptions on D and σ hold. Then, the bisweep data $\varsigma_\sigma : \partial D \times \partial D \to \mathbb{R}$ determine the whole Neumann-to-Dirichlet map Λ_σ.*

Proof. As the considered Neumann-to-Dirichlet maps are self-adjoint, we have

$$\begin{aligned}
2\,&[(\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)](z) - 2\,[(\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)](x) \\
&= \langle(\delta_z - \delta_x), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)\rangle_{\partial D} + \langle(\delta_x - \delta_y), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_z - \delta_x)\rangle_{\partial D} \\
&= \langle(\delta_z - \delta_x), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_z - \delta_y)\rangle_{\partial D} + \langle(\delta_x - \delta_y), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_z - \delta_y)\rangle_{\partial D} \\
&\quad + \langle(\delta_z - \delta_x), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_z)\rangle_{\partial D} + \langle(\delta_x - \delta_y), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_y - \delta_x)\rangle_{\partial D} \\
&= \varsigma_\sigma(z, y) - \varsigma_\sigma(z, x) - \varsigma_\sigma(x, y),
\end{aligned}$$

for any $x, y, z \in \partial D$. Now, we fix x, y and let z vary over all location on ∂D, which means that the left-hand side of the above chain of equalities samples one representative in the quotient equivalence class $2(\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)$, namely the one with grounding at x. Because for each z the corresponding right-hand side is a linear combination of three bisweep data, we deduce that the knowledge of $\varsigma_\sigma : \partial D \times \partial D \to \mathbb{R}$ determines $(\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_x - \delta_y)$ for any $x, y \in \partial D$. On the other hand, the linear span of the set

$$\{\delta_x - \delta_y\}_{x,y\in\partial D} \tag{7}$$

is dense in $H^s_\diamond(\partial D)$ for small enough $s = s_n \in \mathbb{R}$ due to, say, the denseness of $\mathscr{D}_\diamond(\partial D)$ in $H^s_\diamond(\partial D)$, a suitable quadrature rule on ∂D, and the Sobolev embedding theorem for the dual $H^{-s}(\partial D)/\mathbb{C}$. (In fact, the linear span of (7) remains dense if $y \in \partial D$ is fixed). This completes the proof. $\square$

Together with the simple polarization argument in the proof of Theorem 2.1, our main theoretical tool is the fact that in two dimensions ς_σ extends as a complex analytic function of two variables to some (interior) neighborhood of $\partial D \times \partial D$, which will be proven in Sections 3.2 and 3.3. This result leads to the following local characterization of the Neumann-to-Dirichlet map; see Section 3.4 for the proof. Here and in what follows, we denote by

$$\mathfrak{D}_z = \{f \in \mathscr{D}'_\diamond(\partial D) \;:\; \operatorname{supp} f = z \in \partial D\}$$

the subspace of mean-free distributions that are supported at some fixed $z \in \partial D$.

Theorem 2.2. *Let the above assumptions on D and σ hold, and suppose furthermore that $n = 2$. For any fixed $z \in \partial D$, the point measurements of the type*

$$\langle f, (\Lambda_\sigma - \Lambda_{\mathbb{1}})f\rangle_{\partial D}, \qquad f \in \mathfrak{D}_z \tag{8}$$

determine the whole Neumann-to-Dirichlet map Λ_σ.

Since the knowledge of the Neumann-to-Dirichlet map is equivalent to that of the Dirichlet-to-Neumann map, the works of Astala and Päivärinta [2, Theorem 1] and Astala, Lassas and Päivärinta [3, Theorem 1] provide us immediately with the following uniqueness result for the Calderón problem.

Corollary 2.3. *Let the assumptions of Theorem 2.2 hold. Then, the point measurements* (8) *for any fixed $z \in \partial D$ uniquely define an isotropic conductivity σ. If the conductivity σ is anisotropic, the point measurements* (8) *determine σ uniquely up to a pushforward by a boundary-fixing H^1-diffeomorphism of D onto itself.*

Compared to previous local uniqueness results for the Calderón problem, Corollary 2.3 is truly local: the applied current patterns are the distributions supported at a single point and the resulting relative boundary potentials are sampled by these same distributions; cf. [17, Corollary 1.1 and Theorem 1.2]. On the negative side, the assumption that D is two-dimensional and σ equals 1 close to ∂D seems to be inherent in the proof of Theorem 2.2.

Remark 2.4. *Suppose $\Gamma \times \Gamma \subset \partial D \times \partial D$ is a countably infinite set with the accumulation point (z, z). The restriction of the bisweep data ς_σ to $\Gamma \times \Gamma$ determines the point measurements* (8) *and therefore, Theorem 2.2 and Corollary 2.3 become applicable. Indeed, it follows straightforwardly from the polarization identity in the proof of Theorem 2.1 that the assumed measurements define $\langle f, (\Lambda_\sigma - \Lambda_{\mathbf{1}})g\rangle_{\partial D}$ for all $f, g \in \mathfrak{U}_\Gamma$ where*

$$\mathfrak{U}_\Gamma = \operatorname{span}\{\delta_x\}_{x \in \Gamma} \cap \mathscr{D}'_\diamond(\partial D).$$

Since $\mathfrak{D}_z$ belongs to the closure of $\mathfrak{U}_\Gamma$ in the (weak) topology of $\mathscr{D}'(\partial D)$, the point measurements (8) *are determined.*

3. Proof of the main results

3.1. **A factorization of the N-to-D map.** Choose $\Omega \subset \mathbb{R}^n$ to be a simply connected $\mathscr{C}^\infty$-domain, such that $\Sigma \subset \Omega$ and $\overline{\Omega} \subset D$. We define an auxiliary operator

$$A : f \mapsto \frac{\partial u_{\mathbb{1}}}{\partial \nu}\Big|_{\partial\Omega}, \quad \mathscr{D}'_\diamond(\partial D) \to \mathscr{D}_\diamond(\partial\Omega),$$

where $u_{\mathbb{1}}$ is the background solution corresponding to the boundary current pattern f. Notice that $u_{\mathbb{1}}$ is well defined for any $f \in \mathscr{D}'_\diamond(\partial D)$ because each such (compactly supported) current density belongs to $H^s_\diamond(\partial D)$ for some $s = s_f \in \mathbb{R}$, and consequently A is also well defined due to the Gauss divergence theorem and interior elliptic regularity (cf., e.g., [24]). Moreover, A is bounded as a map from $H^s_\diamond(\partial D)$ to $H^{-s}_\diamond(\partial\Omega)$ for any $s \in \mathbb{R}$; see, e.g., [24]. The following factorization can be considered a variant of [12, Theorem 3.1].

Theorem 3.1. *The operator $\Lambda_\sigma - \Lambda_{\mathbb{1}} : \mathscr{D}'_\diamond(\partial D) \to \mathscr{D}(\partial D)/\mathbb{C}$ can be factorized as*

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = A'GA \tag{9}$$

where $G : \mathscr{D}'_\diamond(\partial\Omega) \to \mathscr{D}(\partial\Omega)/\mathbb{C}$ can be interpreted as a bounded map from $H^s_\diamond(\partial\Omega)$ to $H^{-s}(\partial\Omega)/\mathbb{C}$ for any $s \in \mathbb{R}$.

Proof. According to [14, Corollary 3.2], the Neumann-to-Dirichlet map can be factorized as

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = B'FB,$$

where $B : \mathscr{D}'_\diamond(\partial D) \to \mathscr{D}(\partial\Omega)/\mathbb{C}$ maps f to $u_{\mathbb{1}}|_{\partial\Omega}$ and $F : H^{-s+1}(\partial\Omega)/\mathbb{C} \to H^{s-1}_\diamond(\partial\Omega)$ is a bounded operator for any $s \in \mathbb{R}$. Clearly, one can write $B = \lambda A$, where $\lambda : H^{-s}_\diamond(\partial\Omega) \to H^{-s+1}(\partial\Omega)/\mathbb{C}$, the Neumann-to-Dirichlet map for the Laplacian in Ω, is bounded (cf. (5)). Therefore,

$$\Lambda_\sigma - \Lambda_{\mathbb{1}} = A'\lambda' F\lambda A =: A'GA,$$

where the bounded dual operator $\lambda' : H^{s-1}_\diamond(\partial\Omega) \to H^s(\partial\Omega)/\mathbb{C}$ is, in fact, identical to λ, but interpreted as an operator between different Sobolev spaces. Consequently, $G = \lambda' F \lambda : H^{-s}_\diamond(\partial\Omega) \to H^s(\partial\Omega)/\mathbb{C}$ is bounded for any $s \in \mathbb{R}$, which completes the proof. □

In what follows, we interpret G to be a bounded operator from $L^2(\partial\Omega)$ to itself by identifying it with

$$P'GP : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

where $P : L^2(\partial\Omega) \to L^2_\diamond(\partial\Omega)$ is an orthogonal projection and $P' : L^2(\partial\Omega)/\mathbb{C} \to L^2(\partial\Omega)$ is its dual. It is easy to check that P' picks the unique mean-free element of an equivalence class in $L^2(\partial\Omega)/\mathbb{C}$. We continue to denote this newly defined G by the original symbol, and note that the factorization (9) remains valid because the range of A consists of mean-free elements and A' does not 'see' the constant function.

In particular, take note that Theorem 3.1 provides the presentation

$$\varsigma_\sigma(x,y) = \langle A(\delta_x - \delta_y), GA(\delta_x - \delta_y)\rangle_{\partial\Omega}, \qquad x, y \in \partial D \tag{10}$$

for the bisweep data defined originally by (6).

3.2. Holomorphic extension of bisweep data in the unit disk.

In this section, we assume that $D = B \subset \mathbb{R}^2$ is the open unit disk and note that the gradient of the corresponding background solution for the Laplacian $u_{\mathbb{1}}^{z_1,z_2}$, $z_1, z_2 \in \partial D$, with the boundary current density $f = \delta_{z_2} - \delta_{z_1}$ is (cf., e.g., [14])

$$\nabla u_{\mathbb{1}}^{z_1,z_2}(x) = \frac{1}{\pi}\left(\frac{x - z_1}{|x - z_1|^2} - \frac{x - z_2}{|x - z_2|^2}\right), \qquad x \in D. \tag{11}$$

We identify the mapping $(x, z_1, z_2) \mapsto \nabla u_{\mathbb{1}}^{z_1,z_2}(x)$ (from $D \times \partial D^2 \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2$) with a complex function $(\xi, \zeta_1, \zeta_2) \mapsto v(\xi, \zeta_1, \zeta_2)$, which is a map from $D \times \partial D^2 \subset \mathbb{C} \times \mathbb{C}^2$ to $\mathbb{C}$ (cf. [12, 14, 16]). To be more precise,

$$v(\xi, \zeta_1, \zeta_2) = \frac{1}{\pi}\left(\tfrac{\xi - \zeta_1}{(\xi-\zeta_1)\overline{(\xi-\zeta_1)}} - \tfrac{\xi-\zeta_2}{(\xi-\zeta_2)\overline{(\xi-\zeta_2)}}\right) = \frac{1}{\pi}\left(\frac{\zeta_1}{\zeta_1\overline{\xi} - 1} - \frac{\zeta_2}{\zeta_2\overline{\xi} - 1}\right), \tag{12}$$

where we took advantage of the fact that $|\zeta_1|^2 = |\zeta_2|^2 = 1$. For fixed $\xi \in D$, $v(\xi, \cdot, \cdot)$ extends as a holomorphic function to the set $(\mathbb{C} \setminus \{1/\overline{\xi}\}) \times (\mathbb{C} \setminus \{1/\overline{\xi}\}) = (\mathbb{C} \setminus \{1/\overline{\xi}\})^2 \subset \mathbb{C}^2$; let us denote this extension by $w_1(\xi, \cdot, \cdot)$. Analogously, the complex conjugate of v, i.e.,

$$\overline{v(\xi, \zeta_1, \zeta_2)} = \frac{1}{\pi}\left(\frac{1}{\xi - \zeta_1} - \frac{1}{\xi - \zeta_2}\right),$$

can be extended as a holomorphic function, say $w_2(\xi,\cdot,\cdot)$, to $(\mathbb{C}\setminus\{\xi\})^2$. It thus follows that also the real and imaginary parts of $v(\xi,\cdot,\cdot)$ have holomorphic extensions

$$\begin{aligned} v_1(\xi,\zeta_1,\zeta_2) &= \frac{1}{2}\big(w_1(\xi,\zeta_1,\zeta_2)+w_2(\xi,\zeta_1,\zeta_2)\big),\\ v_2(\xi,\zeta_1,\zeta_2) &= \frac{1}{2i}\big(w_1(\xi,\zeta_1,\zeta_2)-w_2(\xi,\zeta_1,\zeta_2)\big) \end{aligned} \tag{13}$$

to $(\mathbb{C}\setminus(\{\xi\}\cup\{1/\overline{\xi}\}))^2$. We denote $V(\xi,\zeta_1,\zeta_2)=[v_1(\xi,\zeta_1,\zeta_2),v_2(\xi,\zeta_1,\zeta_2)]^{\mathrm{T}}$.

Let $U\subset\mathbb{C}$ be an open neighbourhood of ∂D such that $\overline{\Omega}\cap\overline{U}=\emptyset$ and $\overline{\Omega^*}\cap\overline{U}=\emptyset$, where Ω^* is the reflection of Ω with respect to the unit circle ∂D. Due to (10), the bisweep data can be identified with the restriction $\varsigma_\sigma(\zeta_1,\zeta_2)|_{(\partial D)^2}$ of the function $\varsigma_\sigma(\zeta_1,\zeta_2): U^2\to\mathbb{C}$,

$$\varsigma_\sigma(\zeta_1,\zeta_2)=\langle A(\zeta_1,\zeta_2),GA(\zeta_1,\zeta_2)\rangle_{\partial\Omega}=\int_{\partial\Omega}h(\xi,\zeta_1,\zeta_2)(G_y h(y,\zeta_1,\zeta_2))(\xi)\,\mathrm{d}s_\xi, \tag{14}$$

where $\mathrm{d}s_\xi$ corresponds to the (real) arc length measure on $\partial\Omega$ and $A: U^2\to L^2(\partial\Omega)$ is defined by

$$(A(\zeta_1,\zeta_2))(\xi)=h(\xi,\zeta_1,\zeta_2):=\nu_\xi\cdot V(\xi,\zeta_1,\zeta_2),$$

with ν_ξ being the (real) unit normal of $\partial\Omega$ at ξ. We will now show that ς_σ is holomorphic in U^2.

Lemma 3.2. *The operator $A(\zeta_1,\zeta_2)$ is holomorphic in $\zeta_1\in U$ (resp. $\zeta_2\in U$) for an arbitrary fixed value of $\zeta_2\in U$ (resp. $\zeta_1\in U$).*

Proof. Let $M>0$ be a real constant that satisfies

$$\left|\frac{\partial^2}{\partial\zeta_1^2}v_l(\xi,\zeta_1,\zeta_2)\right|\le M$$

for all $\zeta_1,\zeta_2\in U$, $\xi\in\partial\Omega$ and $l=1,2$. Fix arbitrary $\zeta_1\in U$ and let $r>0$ be such that $\{w\in\mathbb{C}: |w-\zeta_1|<r\}\subset U$. By representing the difference $v_l(\xi,\zeta_1+\eta,\zeta_2)-v_l(\xi,\zeta_1,\zeta_2)$ as a complex line integral and subsequently applying the same idea to the derivative of v_l with respect to ζ_1, it follows easily that

$$\left|\frac{v_l(\xi,\zeta_1+\eta,\zeta_2)-v_l(\xi,\zeta_1,\zeta_2)}{\eta}-\frac{\partial v_l}{\partial\zeta_1}(\xi,\zeta_1,\zeta_2)\right|\le\frac{1}{2}M|\eta|,\qquad l=1,2,$$

for all $\xi\in\partial\Omega$, $\zeta_2\in U$ and $0\neq\eta\in\mathbb{C}$ such that $|\eta|<r$. In consequence,

$$\left\|\frac{h(\cdot,\zeta_1+\eta,\zeta_2)-h(\cdot,\zeta_1,\zeta_2)}{\eta}-\frac{\partial h}{\partial\zeta_1}(\cdot,\zeta_1,\zeta_2)\right\|_{L^2(\partial\Omega)}\le M|\eta|\sqrt{|\partial\Omega|},$$

which means that A is holomorphic in ζ_1. The same argument can be applied to ζ_2, and the claim follows. $\square$

Lemma 3.3. *Let X and Y be complex Banach spaces, $\langle\cdot,\cdot\rangle: X\times Y\to\mathbb{C}$ a bounded bilinear form, and $f: U\to X$, $g: U\to Y$ differentiable in an open set $U\subset\mathbb{C}$ (resp. $U\subset\mathbb{R}$). Then,*

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle f(z),g(z)\rangle=\langle f'(z),g(z)\rangle+\langle f(z),g'(z)\rangle$$

for any $z\in U$. In particular, if f and g are holomorphic, then so is the map $U\ni z\mapsto\langle f(z),g(z)\rangle\in\mathbb{C}$.

Proof. The assertion follows from essentially the same argument as the standard product rule of calculus. $\square$

Theorem 3.4. *The bisweep data extend to a holomorphic function* $\varsigma_\sigma : U^2 \to \mathbb{C}$, *where* U *is an open neighborhood of* $\partial D \subset \mathbb{C}$.

Proof. We have interpreted G as a bounded operator from $L^2(\partial\Omega)$ to itself, which makes

$$(p, q) \mapsto \langle p, Gq \rangle_{\partial\Omega} = \int_{\partial\Omega} p(\xi)(Gq)(\xi)\, \mathrm{d}s_\xi$$

a bounded bilinear form on $L^2(\partial\Omega) \times L^2(\partial\Omega)$. By (14) and Lemmas 3.2 and 3.3, the extension $\varsigma_\sigma : U^2 \to \mathbb{C}$ is holomorphic in either variable if the other has an arbitrary fixed value. Due to the Hartogs' theorem [15, Theorem 2.2.8], this means that ς_σ is, in fact, analytic in U^2, that is, it locally coincides with its multi-dimensional complex Taylor series. □

Corollary 3.5. *The angular bisweep data* $\tilde{\varsigma}_\sigma : \mathbb{R}^2 \to \mathbb{R}$,

$$\tilde{\varsigma}_\sigma(\theta_1, \theta_2) := \varsigma_\sigma(e^{i\theta_1}, e^{i\theta_2}), \tag{15}$$

is an analytic function.

Proof. Clearly, the definition (15) can be extended to some open set $V^2 \subset \mathbb{C}^2$ such that $\mathbb{R} \subset V$ and $e^{iV} \subset U$, and by the chain rule, it is a holomorphic function of two variables. As the restriction of such a function to $\mathbb{R}^2$, the angular sweep data is analytic, that is, locally coincides with its two-dimensional real Taylor series. □

In what follows, we denote by D_j the derivative with respect to the jth variable. Due to the theory of analytic continuation, we have

Corollary 3.6. *The (angular) bisweep data is completely determined by the set of its derivatives*

$$\{D_1^j D_2^k \tilde{\varsigma}_\sigma(\theta_1, \theta_2) \ : \ j, k \in \mathbb{N}_0\}$$

at an arbitrary point $(\theta_1, \theta_2) \in \mathbb{R}^2$.

3.3. **Generalization to smooth domains.** Let us then adopt the setting of Theorem 2.2 and assume, in particular, that $D \subset \mathbb{R}^2$ is a simply connected and bounded domain with a $\mathscr{C}^\infty$-boundary.

Let Φ be a conformal map of the open unit disk B onto D. According to [27, Section 3.3], $\Phi|_{\partial B}$ in turn defines a smooth diffeomorphism of ∂B onto ∂D. In the spirit of (15), we introduce the angular bisweep data $\tilde{\varsigma}_\sigma : \mathbb{R}^2 \to \mathbb{R}$,

$$\tilde{\varsigma}_\sigma(\theta_1, \theta_2) := \varsigma_\sigma\big(\Phi(e^{i\theta_1}), \Phi(e^{i\theta_2})\big), \tag{16}$$

and generalize Corollary 3.6 to our new framework.

Corollary 3.7. *Assume that* D *satisfies the above assumptions. Then, the angular bisweep data* (16) *is completely determined by the set of its derivatives*

$$\{D_1^j D_2^k \tilde{\varsigma}_\sigma(\theta_1, \theta_2) \ : \ j, k \in \mathbb{N}_0\} \tag{17}$$

at an arbitrary point $(\theta_1, \theta_2) \in \mathbb{R}^2$.

Proof. It follows directly from the argument in the proof of [11, Theorem 3.2] that

$$\varsigma_{\sigma^*} := \varsigma_\sigma(\Phi|_{\partial B}(\cdot), \Phi|_{\partial B}(\cdot)) : \partial B \times \partial B \to \mathbb{R} \tag{18}$$

is the bisweep data corresponding to the unit disk and the pull-back conductivity

$$\sigma^* := J_\Phi^{-1}(\sigma \circ \Phi)(J_\Phi^{-1})^{\mathrm{T}} \det J_\Phi,$$

where J_Φ is the Jacobian matrix of Φ (interpreted as a map from $\mathbb{R}^2$ to itself). Notice that σ^* is a feasible conductivity, i.e., it is strictly positive definite and equals one in some interior neighborhood of ∂B, due to the basic properties of conformal mappings (cf. [11, Section 3]).

By definition,

$$\tilde{\varsigma}_{\sigma^*} = \tilde{\varsigma}_\sigma$$

where $\tilde{\varsigma}_{\sigma^*}$ is the angular sweep data for σ^* on ∂B defined by (15) and $\tilde{\varsigma}_\sigma$ is given by (16). According to Corollary 3.6, $\tilde{\varsigma}_{\sigma^*}$ is determined by the set of its derivatives (17), which completes the proof. □

Remark 3.8. *The smoothness of ∂D is needed for Corollary 3.7 so that the definition* (6) *makes sense — for which less regularity would certainly suffice — and that the assumptions of* [11, Theorem 3.2] *are satisfied. In consequence, if* [11, Theorem 3.2] *extends to more general domains with less regular boundaries (as it does), Corollary 3.7 adopts the corresponding regularity assumptions on ∂D. This same reduction of smoothness carries over to Theorem 2.2 and Corollary 2.3 with the extra requirement of* local *$\mathscr{C}^\infty$-smoothness around $z \in \partial D$, as apparent from the proof presented in the following section.*

3.4. **Uniqueness by point measurements.** Let us adopt the assumptions and the notation of Section 3.3. We will prove the claim of Theorem 2.2 by utilizing the pointwise-supported mean-free distributions $\{\delta_\theta^{(j)}\}_{j=1}^\infty \subset \mathfrak{D}_{\Phi(e^{i\theta})}$ defined via

$$\langle \delta_\theta^{(j)}, \varphi \rangle_{\partial D} = \frac{\mathrm{d}^j}{\mathrm{d}\vartheta^j} \varphi(\Phi(e^{i\vartheta}))|_{\vartheta=\theta}, \qquad \varphi \in \mathscr{D}(\partial D), \quad j \in \mathbb{N}.$$

We extend this definition in the natural way to the case $j = 0$ by requiring that $\langle \delta_\theta^{(0)}, \varphi \rangle_{\partial D} = \langle \delta_\theta, \varphi \rangle_{\partial D} = \varphi(\Phi(e^{i\theta}))$, but note that this standard (angular) delta distribution is not mean-free. It follows easily from the Sobolev embedding theorem that the mapping $\theta \mapsto \delta_\theta^{(k)}$ is differentiable, say, from $\mathbb{R}$ to $H^{-k-3}(\partial D)$, $k \in \mathbb{N}_0$, and that the corresponding derivative is $\theta \mapsto \delta_\theta^{(k+1)}$.

The angular bisweep data of (16) allows the representation

$$\tilde{\varsigma}_\sigma(\theta_1, \theta_2) = \langle (\delta_{\theta_1} - \delta_{\theta_2}), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_{\theta_1} - \delta_{\theta_2}) \rangle_{\partial D}$$

for any $\theta_1, \theta_2 \in \mathbb{R}$. Obviously,

$$\tilde{\varsigma}_\sigma(\theta, \theta) = 0$$

for any $\theta \in \mathbb{R}$. Due to Lemma 3.3 and the boundedness (and self-adjointness) of $\Lambda_\sigma - \Lambda_{\mathbb{1}} : H_\diamond^s(\partial D) \to H^{-s}(\partial D)/\mathbb{C}$ for any $s \in \mathbb{R}$, we have

$$D_1 \tilde{\varsigma}_\sigma(\theta_1, \theta_2) = 2\langle \delta_{\theta_1}^{(1)}, (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_{\theta_1} - \delta_{\theta_2}) \rangle, \tag{19}$$

which vanishes for $\theta_1 = \theta_2 = \theta$. Similarly, the kth derivative of the angular bisweep data with respect to the first variable at (θ_1, θ_2) reads

$$D_1^k \tilde{\varsigma}_\sigma(\theta_1, \theta_2) = 2\langle \delta_{\theta_1}^{(k)}, (\Lambda_\sigma - \Lambda_{\mathbb{1}})(\delta_{\theta_1} - \delta_{\theta_2}) \rangle + \sum_{j=1}^{k-1} \binom{k}{j} \langle \delta_{\theta_1}^{(j)}, (\Lambda_\sigma - \Lambda_{\mathbb{1}}) \delta_{\theta_1}^{(k-j)} \rangle, \tag{20}$$

meaning that

$$D_1^k \tilde{\varsigma}_\sigma(\theta, \theta) = \sum_{j=1}^{k-1} \binom{k}{j} \langle \delta_\theta^{(j)}, (\Lambda_\sigma - \Lambda_{\mathbb{1}}) \delta_\theta^{(k-j)} \rangle, \qquad \theta \in \mathbb{R}, \ k \geq 2.$$

It clearly holds that $D_2^k\tilde{\varsigma}_\sigma(\theta,\theta) = D_1^k\tilde{\varsigma}_\sigma(\theta,\theta)$ for any $\theta \in \mathbb{R}$ and $k \in \mathbb{N}$. Moreover, taking the lth derivative of (19) and (20) with respect to the second variable results in

$$D_1^k D_2^l \tilde{\varsigma}_\sigma(\theta_1,\theta_2) = -2\langle \delta_{\theta_1}^{(k)}, (\Lambda_\sigma - \Lambda_{\mathbb{1}})\delta_{\theta_2}^{(l)}\rangle, \qquad \theta_1,\theta_2 \in \mathbb{R},\ k,l \in \mathbb{N}.$$

In consequence, we have altogether shown that any partial derivative of the angular bisweep data at $\theta_1 = \theta_2 = \theta \in \mathbb{R}$ is either known to vanish or can be given as a linear combination of terms of the form

$$\langle f, (\Lambda_\sigma - \Lambda_{\mathbb{1}})g\rangle_{\partial D}, \qquad f,g \in \mathfrak{D}_z,$$

where $z = \Phi(e^{i\theta})$ can be chosen arbitrarily via the choice of θ. Due to the standard polarization identity

$$4\langle f, (\Lambda_\sigma - \Lambda_{\mathbb{1}})g\rangle_{\partial D} = \langle (f+g), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(f+g)\rangle_{\partial D} - \langle (f-g), (\Lambda_\sigma - \Lambda_{\mathbb{1}})(f-g)\rangle_{\partial D},$$

this means that the knowledge of the point measurements (8) for all $f \in \mathfrak{D}_z$ (with fixed $z \in \partial D$) implies the knowledge of all the derivatives (17) at the corresponding polar angle $\theta_1 = \theta_2 = \theta$.

The statement of Theorem 2.2 follows now from Corollary 3.7 and Theorem 2.1.

Acknowledgements

Nuutti Hyvönen and Petteri Piiroinen would like to thank John Sylvester, Bastian Harrach, Lauri Oksanen, Roland Griesmaier and Martin Simon for the discussions about two-electrode measurements at the Oberwolfach workshop on Inverse Problems for Partial Differential Equations (ID: 1208b) organized by Martin Hanke, Andreas Kirsch, William Rundell and Matti Lassas.

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